Q.2 a. A Computer company requires 30 programmers to handle system programming Jobs and 40 programmers for application programming. If the company appoints 55 programmers to carry out these Jobs, how many of these perform Job of both types ? How many can handle only system programming jobs? How many can handle application programming.

b. Three students x, y, z write an examination. Their chances of passing are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that.

(i) all of them pass (ii) atleast one of them passes.

Answer:

a) Let A → Set of programmers who handle system programming Job.
B → Set of programmers who handle application programming.
Given $|A| = 30$: $|B| = 40$; $|A \cup B| = 55$.
Addition rule is $|A \cup B| = |A| + |B| - |A \cap B|$
$|A \cap B| = |A| + |B| - |A \cup B| = 15$

4 marks

This means 15 programmers perform both types of Jobs
∴ The number of programmers who handle only system programming Job is $|A - B| = |A| - |A \cap B| = 30 - 15 = 15$
and the number of programmers who handle only application programming is $|B - A| = |B| - |A \cap B| = 40 - 15 = 25$.

4 marks

b) Define X ← event that the student x passing the examination
Y ← ——— , ... y ———
Z ← ——— z ———

Given $P(X) = \frac{1}{2}$: $P(Y) = \frac{1}{3}$: $P(Z) = \frac{1}{4}$.
$P(\bar{X}) = \frac{1}{2}$: $P(\bar{Y}) = \frac{2}{3}$: $P(\bar{Z}) = 3/4$.
Let E_1 ← event that all of them pass
∴ $P(E_1) = P$[all three of them passing the examination]
$= P[X \cap Y \cap Z] = P(X)P(Y)P(Z) = (\frac{1}{2})(\frac{1}{3})(\frac{1}{4}) = \frac{1}{24}$.

4 marks

Let E_2 ← event that atleast one of them passing the examination
$P(E_2) = 1 - P$(none of them passing the Examination)
$= 1 - P(\bar{X} \cap \bar{Y} \cap \bar{Z}) = 1 - P(\bar{X})P(\bar{Y})P(\bar{Z}) = 3/4$.

4

Q.3 b. Show that $\neg\forall x\ [P(x)\rightarrow Q(x)]$ and $\exists x[P(x)\wedge\neg Q(x)]$ are logically equivalent.

Answer:

Let $\neg\forall x\,(P(x)\rightarrow Q(x))$ is true

$\Leftrightarrow$ $\forall x\,(P(x)\rightarrow Q(x))$ is False

$\Leftrightarrow$ $(P(x)\rightarrow Q(x))$ is False

$\Leftrightarrow$ $P(x)$ is true and $Q(x)$ is False for every x in the domain.

$\Leftrightarrow$ $P(x)$ is true for all x in the domain and $Q(x)$ is False for some x in the domain

$\Rightarrow$ $P(x)$ is true for all x in the domain and $\neg Q(x)$ is true for some x in the domain.

$\Rightarrow$ $(P(x)\wedge\neg Q(x))$ is true for some x in the domain.

$\Leftrightarrow$ $\exists x\,(P(x)\wedge\neg Q(x))$ is true

$\therefore$ $\neg\forall x\,(P(x)\rightarrow Q(x))\equiv\exists x\,(P(x)\wedge\neg Q(x))$.

8 marks

Q.4 a. State any Four Rules of Inference and explain.

Answer:

The Four rules of Inference are

(i) Rule of conjunctive simplification: If p and q are any two propositions and if $p\wedge q$ is true, then p is true.

ie $(p\wedge q)\Rightarrow p$. 02 m

(ii) Rule of Disjunctive Amplification: If p and q are any two propositions and if p is true, then $p\vee q$ is true ie $p\Rightarrow(p\vee q)$. 02 marks

(iii) Rule of Syllogism: If p, q and r are any three propositions and if $p\rightarrow q$ is true and $q\rightarrow r$ is true, then $p\rightarrow r$ is true

ie
$$\begin{array}{l} p\rightarrow q \\ q\rightarrow r \\ \hline \therefore\ p\rightarrow r \end{array}$$
02 marks

(iv) Modus ponens (Rule of Detachment): If p is true and $p\rightarrow q$ is true then q is true ie
$$\begin{array}{c} p \\ p\rightarrow q \\ \hline \therefore\ q \end{array}$$
02 marks.

(v) Modus Tollen's:

If $p\rightarrow q$ is true and q is False, then p is False (from Four)

ie
$$\begin{array}{c} p\rightarrow q \\ \neg q \\ \hline \therefore\ \neg p \end{array}$$

(Note: Any Four can be considered)

b. Show that the hypothesis" If you send me an e-mail message, then I will finish writing the program" "If you don't send me an e-mail message, then I will go to sleep early". And If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion " If I do not finish writing the program, then I will wake up feeling refreshed.

Answer:

Let p, q, r, s be the propositions
p ← "you send me an e-mail message"
q ← " I will finish writing the program"
r ← " I will go to sleep early."
s ← " I will wake up feeling refreshed".
Then the hypothesis are $p \to q$
$\neg p \to r$ and
$r \to s$
conclusion is $\neg q \to s$.

4 marks for making the hypothesis

4 marks for deducing the conclusion.

Step	Reason
1. $p \to q$	Hypothesis
2. $\neg q \to \neg p$	Contrapositive of 1
3. $\neg p \to r$	Hypothesis
4. $\neg q \to r$	Hypothetical syllogism using 2 and 3.
5. $r \to s$	Hypothesis
6. $\neg q \to s$	Hypothetical syllogism using 4 and 5

Q.5 a. Prove the following statement by mathematical induction. If a set has n elements then its powerset has 2^n elements.

Answer:

If A is any set with $|A| = n$ then $|P(A)| = 2^n$

If $n = 0$, then $|A| = 0$ then $|P(A)| = 2^0 = 1$, true.

i.e $P(A) = \{\phi\}$.

If A is a set with $|A| = K$, then $|P(A)| = 2^K$ (ie A has 2^K subsets)

If B is a set with $|B| = K+1$, we shall prove that $|P(B)| = 2^{K+1}$ (B has 2^{K+1} subsets) Define a set $C = B - \{x\}$, where x is any particular element. Then $|C| = K$ $\therefore |P(C)| = 2^K$.

ie there are 2^K subsets of C which are also subsets of B. Take the union of all these subsets with $\{x\}$ which gives another 2^K subsets of B. Thus the total subsets of B $= 2^K + 2^K = 2^{K+1}$.

The result is proved for $n = K+1$ also. Hence it is true for all integral values of n.

8 marks

b. Suppose U is a universal set and $A, B_1, B_2 ......B_n \subseteq U$ prove that $A \cap (B_1 \cup B_2 \cup .... \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup ....(A \cap B_n)$

Answer:

By distributive Law we have $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$

Result is true for $n = 2$

We shall assume that the result is true for $n = K \geq 2$ ie

$A \cap (B_1 \cup B_2 \cup .... \cup B_K) = (A \cap B_1) \cap (A \cap B_2) \cap ..... \cap (A \cap B_K)$

We shall now prove the result for $n = K+1$, consider.

$A \cap (B_1 \cup B_2 \cup ..... \cup B_{K+1}) = A \cap \{(B_1 \cup B_2 \cup ..... \cup B_K) \cup B_{K+1}\}$

$= \{A \cap (B_1 \cup B_2 \cup ..... \cup B_K)\} \cup (A \cap B_{K+1})$ 8 we

$= (A \cap B_1) \cup (A \cap B_2) \cup ..... (A \cap B_K) \cup (A \cap B_{K+1})$

This shows that the result is true for $n = K+1$ also. Hence by Mathematical induction the result is true for $n \geq 2$.

Q.6 a. Define the following (i) Reflexive (ii) Symmetric (iii) Transitive properties of Relation with an example. What is an equivalence relation?

Answer:

Reflexive property: A Relation R on a set A is said to be Reflexive if $\forall a \in A$, $(a, a) \in R$. 2

Ex: Let N represents the set of Natural numbers and R is the relation on N such that $(a, b) \in R$ if a/b, we know that a/a $\therefore (a, a) \in R$ $\forall a \in N$ $\therefore$ R is Reflexive.

Symmetric property: A Relation R on a set A is said to be symmetric if $\forall a, b \in A$ and $(a, b) \in R \Rightarrow (b, a) \in R$ 2

Ex: Let N be the set of Natural numbers and R is the relation on N such that $a, b \in R$ if $a-b$ is a multiple of 5 $\Rightarrow (b-a)$ is also a multiple of 5.

Transitive property: A Relation R on a set A is said to be Transitive if $\forall a, b, c \in A$ whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ 2

Ex: Let N be the set of Natural numbers and R is the relation on N $\ni$ $(a, b) \in R$ if '$a < b$'. If $a < b$ and $b < c$ then $a < c$ $\therefore$ R is Tr.

b. Define the partial order and POSET. If R is a Relation on the set A = {1,2,3,4,} defined by $R = \{(x, y) | x, y \in A \text{ and } x \text{ divides } y\}$ prove that (A, R) is a POSET and draw its Hasse diagram.

Answer:

Let A be a nonempty set and R is a relation defined on A such that (i) R is Reflexive (ii) R is antisymmetric (iii) R is Transitive. Then R is said to be a partial order on A.
A set A with a partial order R defined on it is called poset and it is denoted as (A, R). (2)

From the definition of R, we have
$R = \{(x,y) / x,y \in A$ and x divides $y\}$
$= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$.
We observe that $(a,a) \in R$ for all $a \in A$. Hence R is Reflexive on A.
We verify that the elements of R are such that if $(a,b) \in R$ and $a \neq b$ then $(b,a) \notin R$. ∴ R is antisymmetric on A.
Further we check that the elements of R are such that if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ ∴ R is Transitive on A.
Hence R is a partial order on A ∴ (A, R) is a poset. The Hasse diagram for R is as shown below. 6

Q.7 a. If 'o' is an operation on z defined by x o y = x + y +1 prove that (z, o) is an abelian group.

b. Prove that any two left (or right) cosets of a sub group H of a group G are either disjoint or identical.

Answer:

Q 7 a) $\forall\, a,b \in Z$, $a+b+1$ is also an integer $\in Z$
∴ closure axiom is satisfied.
$\forall\, a,b,c \in Z$, $a*(b*c) = a\circ(b\circ c) = a\circ(b+c+1) = a+b+c+2$
$(a\circ b)\circ c = (a+b+1)\circ c = a+b+c+2$
Associative axiom is satisfied.
$\forall\, a \in Z$ we have $a\circ e = a$ ie $a+e+1 = a \Rightarrow e = -1 \in Z$
∴ Identity element is -1 8 m
$\forall\, a \in Z\ \exists\, b \in Z \ni a*b = e \Rightarrow a+b+1 = e$
ie $a+b+1 = -1 \Rightarrow b = -2-a \in Z$
Inverse exists for each element of Z ∴ Inverse axiom is satisfied.

b) Let aH, bH are the two cosets of H
If $aH \cap bH = \phi$ (must prove that $aH = bH$)
Let $c \in aH \cap bH \Rightarrow c \in aH$ and $c \in bH \Rightarrow c = ah_1 : c = bh_2$, $h_1,h_2 \in H$.
∴ $ah_1 = bh_2$ ∴ $a = bh_2h_1^{-1} = bh_3 \rightarrow (1)$
|||ly $b = ah_4$. 8 m
Let $x = ah$ where $h \in H$ ∴ $x = bh_3h$ from (1)
$= bh_5 \in bH \Rightarrow aH$ is a subset of bH
|||ly we can prove that bH is a subset of aH ∴ $aH = bH$.

Q.8 a. If f: $A \to B$ and g : $B \to C$ are Bijective functions then prove that $(gof)^{-1} = f^{-1} o g^{-1}$.

Answer:

Since f and g are bijective functions
$(gof): A \to C$ is also bijective
f and g are bijective functions
$\Rightarrow f^{-1}: B \to A$; $g^{-1}: C \to B$ are also bijective.
$(gof)^{-1}: C \to A$ is also bijective.
Now for $b \in B$ and $c \in C$ $g(b) = c \Rightarrow b = g^{-1}(c)$
$f: A \to B \Rightarrow f(a) = b$, $a \in A$ and $b \in B \Rightarrow a = f^{-1}(b)$
$(gof): A \to C \Rightarrow (gof)(a) = c$ for $a \in A$ and $c \in C$
$a = (gof)^{-1}(c) \to (1)$
consider $(f^{-1} o g^{-1}) c = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a \to (2)$
From (1) and (2) $(gof)^{-1} = f^{-1} o g^{-1}$.

b. If A = B = C = R, the set of all real numbers. Let $f: A \to B$; $g: B \to C$ & f(a) = 2a+1 ; g(b) = b/3 find (i) f o g(-2) (ii) g o f(-1) (iii) verify that $(g \, o \, f)^{-1} = f^{-1} \, o \, g^{-1}$

Answer:

(i) $f o g(-2) = f(g(-2)) = f(-2/3)$ $\quad \because g(b) = b/3$
$= 2(-2/3) + 1 = -1/3$.
(ii) $g o f(-1) = g(f(-1)) = g(-1) = -1/3$.
(iii) $gof: A \to C$
$(gof)(a) = c$ where $a \in A$ and $c \in C$
$g[f(a)] = c \Rightarrow g[2a+1] = c \Rightarrow \frac{2a+1}{3} = c. \Rightarrow a = \frac{3c-1}{2}$
$(gof) a = c \Rightarrow a = (gof)^{-1} c \Rightarrow \frac{3c-1}{2} = (gof)^{-1} c \to (1)$
$f: A \to B$, $f(a) = b$ where $a \in A$ and $b \in B$
$f(a) = 2a+1 = b \Rightarrow a = \frac{b-1}{2}$. $f(a) = b \Rightarrow a = f^{-1}(b) \therefore \frac{b-1}{2} = f^{-1}(b)$
$g: B \to C$, $g(b) = c$ where $b \in B$ $c \in C$
$g(b) = b/3 = c \Rightarrow b = 3c$; $g(b) = c \Rightarrow b = g^{-1}(c) \Rightarrow 3c = g^{-1}(c)$.
consider $(f^{-1} o g^{-1}) c = f^{-1}[g^{-1}(c)] = f^{-1}(3c) = \frac{3c-1}{2}$
$\therefore (gof)^{-1} c = (f^{-1} o g^{-1}) c$.

Q.9 a. The parity – check Matrix for an encoding function E : $Z_2^3 \to Z_2^6$ is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(i) Determine the associated generator Matrix

(ii) Does this code correct all single errors is transmission.

Answer:

We have (i) $H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

which is of the form $[A^T / I_3]$, Accordingly

$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence the associated generator Matrix is

$G = [I_3, A] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

6 marks

(ii) We observe that two columns of H (2^{nd} and 5^{th}) are identical Therefore H doesnot provide a decoding scheme that corrects single errors in transmission.

b. Define a Ring.

Find all integers k and m for which $(z, \oplus, \Theta)$ is a Ring under the binary operations

$x \oplus y = x + y - k$, $x \,\Theta\, y = x + y - m\,x\,y$

Answer:

Definition : Let R be a non empty set which is closed under two binary operations '+' and '•'. Then R together with these operations is called a Ring provided the following axioms hold:-

(i) R is an abelian group under '+'

(ii) The operation '•' is associative in R ie $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ $\forall a,b,c \in R$

(iii) The operation • is distributive over the operation + in R ie

$a\cdot(b+c) = a\cdot b + a\cdot c$; $(a+b)\cdot c = a\cdot c + b\cdot c$ $\forall a,b,c \in R$.

For $(z, \oplus, \odot)$ to be a Ring, it is necessary that the distributive laws must hold (with the other laws). Thus we should have

$x \odot (y \oplus z) = (x \odot y) + (x \odot z)$, By using the definition of $\oplus$ and $\odot$, we find

$x \odot (y \oplus z) = x + (y \oplus z) - mx(y \oplus z) = x + (y+z-k) - mx(y+z-k)$

$= x + y + z - m(xy + xz) - k + mkx \rightarrow (i)$

$(x \odot y) \oplus (x \odot z) = (x \odot y) + (x \odot z) - k$

$= (x + y - mxy) + (x + z - mxz) - k$

$= x + y + z - m(xy + xz) - k + x \rightarrow (ii)$

From (i) & (ii) $x = mkx \Rightarrow mk = 1$

$\therefore$ $m = k = 1$ or $m = k = -1$.

6 marks